Mathematics Assignment: Probability and Equations

Name: ________________________ Class: ___________
Roll No: ___________ Date: ___________

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Q.1 Probability Theory (20)

(a) Find the sample space for choosing an odd number from 1 to 15 at random.

Solution (detailed):

The sample space S is the set of all possible outcomes. We are choosing an odd number from 1 to 15.

Odd numbers between 1 and 15 inclusive are:

1, 3, 5, 7, 9, 11, 13, 15.

Therefore the sample space is:

S = {1, 3, 5, 7, 9, 11, 13, 15}.

Number of elements in S = 8.

(b) What is the difference between mutually exclusive events and collectively exhaustive events?

Solution (detailed):

Definitions:

- Mutually exclusive events: Two events A and B are mutually exclusive if they cannot occur together; that is $A \cap B = \varnothing$ (the intersection is empty).

Example: When tossing a fair coin, events A = {Head} and B = {Tail} are mutually exclusive since a single toss cannot be both head and tail.

- Collectively exhaustive events: A collection of events is collectively exhaustive if at least one of the events must occur; their union equals the sample space.

Example: For a single coin toss, events {Head} and {Tail} are collectively exhaustive because $\{Head\} \cup \{Tail\} = S$, the sample space.

Key difference:

- Mutually exclusive relates to pairwise non-overlap (no simultaneous occurrence).

- Collectively exhaustive relates to covering all possible outcomes (their union is the entire sample space).

Events can be both (like coin toss Head and Tail are mutually exclusive and together exhaustive). They can also be neither.

(c) The probability that an applicant for pilot school will be admitted is 0.5. If three applicants are selected at random, what is the probability that:

i. All three will be admitted

ii. None will be admitted

iii. Only one will be admitted

Solution (detailed):

Let p = 0.5 be the probability a single applicant is admitted, and assume independent selections.

We can use the binomial model: For $n = 3$ applicants, probability of exactly k admitted is

$P(X = k) = C(n, k) * p^k * (1 - p)^{(n - k)}$.

i) All three admitted => $k = 3$.

$P(\text{all three}) = p^3 = (0.5)^3 = 0.125 = 1/8$.

ii) None admitted => $k = 0$.

$P(\text{none}) = (1 - p)^3 = (0.5)^3 = 0.125 = 1/8$.

iii) Only one admitted => $k = 1$.

$P(\text{only one}) = C(3,1) * p^1 * (1-p)^2 = 3 * (0.5) * (0.5)^2 = 3 * (0.5)^3 = 3/8 = 0.375$.

(You can also list all 8 equally likely outcomes H/T patterns and count the favourable ones.)

Q.2 Random Variables (20)

(a) Define a random variable. What is the difference between a discrete random variable and a continuous random variable?

Solution (detailed):

Definition:

A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

Types:

- Discrete random variable: takes countable values (finite or countably infinite), e.g., number of heads in 3 coin tosses, number of customers arriving in an hour.

- Continuous random variable: takes values in a continuum (an interval of real numbers), e.g., the exact waiting time in minutes, the height of a person.

Key differences:

- Discrete: probability mass function (PMF) $P(X = x)$ defined for each possible x. Probabilities are sums.

- Continuous: probability density function (PDF) $f(x)$ exists; probability of any exact point $P(X = x) = 0$; probabilities are areas under the PDF (integrals).

(b) The fire chief for a small volunteer fire department has compiled data on the number of false alarms called in each day for the past 360 days. Construct the probability distribution for this study.

Solution (detailed and method):

To construct the empirical probability distribution:

1. Count how many days had 0 false alarms, 1 false alarm, 2 false alarms, and so on.

2. Form a frequency table: for each count k, let frequency $f(k)$ be the number of days with k false alarms.

3. Convert frequencies to probabilities by $p(k) = f(k) / 360$ (since there are 360 days total).

4. Check that $\sum_k p(k) = 1$ (or very close, rounding aside).

Example (illustrative — since raw data is not provided we use a realistic sample):

Suppose the counts were:

- 0 false alarms: 240 days

- 1 false alarm : 80 days

- 2 false alarms: 30 days

- 3 false alarms: 8 days

- 4 false alarms: 2 days

Total = 240 + 80 + 30 + 8 + 2 = 360 days.

Probability distribution:

k f(k) p(k) = f(k)/360 Decimal approx

0 240 240/360 = 2/3 0.6667

1 80 80/360 = 2/9 0.2222

2 30 30/360 = 1/12 0.08333

3 8 8/360 = 1/45 0.02222

4 2 2/360 = 1/180 0.00556

Check: Sum of probabilities = 0.6667 + 0.2222 + 0.08333 + 0.02222 + 0.00556 ≈ 1.0000.

Interpretation:

This empirical distribution estimates the probability that on a randomly chosen day the department receives k false alarms. (Replace the example frequencies by the actual counts if you have them; the method is the same.)

(c) Construct the discrete probability distribution that corresponds to tossing a fair coin three times. Suppose the random variable X equals the number of heads occurring in three tosses. What is the probability of two or more heads?

Solution (detailed):

Sample space for three coin tosses has 2^3 = 8 equally likely outcomes:

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

Random variable X = number of heads, possible values: 0, 1, 2, 3.

Count occurrences:

- $X = 0$: {TTT} => 1 outcome => $P(X=0) = 1/8$.

- $X = 1$: {HTT, THT, TTH} => 3 outcomes => $P(X=1) = 3/8$.

- $X = 2$: {HHT, HTH, THH} => 3 outcomes => $P(X=2) = 3/8$.

- $X = 3$: {HHH} => 1 outcome => $P(X=3) = 1/8$.

Probability of two or more heads: $P(X >= 2) = P(X=2) + P(X=3) = 3/8 + 1/8 = 4/8 = 1/2 = 0.5$.

Q.3 Equations (20)

Solve the following first-degree equations:

(a) $8x – 6 = 5x + 3$.

Solution (step-by-step):

$8x - 6 = 5x + 3$

Bring x terms to left and constants to right:

$8x - 5x = 3 + 6$

$3x = 9$

$x = 9 / 3 = 3$.

Answer: $x = 3$.

(b) $–15 + 35x = 8x – 9$.

Solution:

$-15 + 35x = 8x - 9$

Bring x terms to left: $35x - 8x = -9 + 15$

$27x = 6$

$x = 6 / 27 = 2 / 9$.

Answer: $x = 2/9$.

(c) $(x + 9) – (–6 + 4x) + 4 = 0$.

Solution:

Expand and simplify carefully:

$(x + 9) - (-6 + 4x) + 4 = 0$

$(x + 9) + 6 - 4x + 4 = 0$ [because minus sign distributes: $-(-6) = +6$; $-(4x) = -4x$]

Combine like terms:

$x - 4x + (9 + 6 + 4) = 0$

$-3x + 19 = 0$

$-3x = -19$

$x = 19 / 3$.

Answer: $x = 19/3$.

Q.4 Equation (20)

(a) Solve the following quadratic equations using the quadratic formula:

i. $4x^2 + 3x - 1 = 0$

Solution (detailed):

Quadratic formula: $x = [-b \pm \sqrt{b^2 - 4ac}] / (2a)$.

Here $a = 4$, $b = 3$, $c = -1$.

Discriminant $D = b^2 - 4ac = 3^2 - 4(4)(-1) = 9 + 16 = 25$.

$\sqrt{D} = 5$.

$x = [-3 \pm 5] / (2*4) = [-3 \pm 5] / 8$.

So two solutions:

$x1 = (-3 + 5) / 8 = 2 / 8 = 1/4$.

$x2 = (-3 - 5) / 8 = (-8) / 8 = -1$.

Answer: $x = 1/4$ or $x = –1$.

ii. $4t^2 – 64 = 0$

Solution:

Simplify: $4t^2 – 64 = 0 \Rightarrow$ divide both sides by 4 $\Rightarrow t^2 – 16 = 0 \Rightarrow t^2 = 16 \Rightarrow t = \pm 4$.

Answer: $t = 4$ or $t = –4$.

(b) A railing is to enclose a rectangular area of 1800 square feet. The length of the plot is twice the width. How much railing must be used?

Solution (detailed):

Let width = w (feet). Then length = 2w (feet).

Area A = length × width = (2w) × w = $2w^2$.

Given area $2w^2 = 1800 \Rightarrow w^2 = 900 \Rightarrow w = 30$ ft (take the positive root for length/width).

Length = 2w = 60 ft.

Perimeter P = 2(length + width) = 2(60 + 30) = 2(90) = 180 ft.

So railing required = 180 feet.

Answer: 180 ft.

Q.5 Linear Equations (20)

(a) Solve the linear equation $y = 2x + 1$.

Solution (detailed):

This equation is already solved for y (slope-intercept form).

- Slope (m) = 2.

- y-intercept (b) = 1 (point (0,1)).

If you need x in terms of y: $x = (y – 1) / 2$.

Example: If $y = 0$ (find x-intercept): $0 = 2x + 1 \Rightarrow 2x = -1 \Rightarrow x = -1/2$.

Graphically: a straight line with slope 2 and intercept 1.

(b) A company has fixed costs of $7,000 for plant and equipment and variable costs of $600 for each unit of output. What is the total cost at varying levels of output?

Solution (detailed):

Let q = number of units produced.

Fixed cost FC = $7,000 (incurred even when q = 0).

Variable cost per unit v = $600, so variable cost = 600*q.

Total cost function: $C(q) = FC + v*q = 7000 + 600q$.

Examples:

- $q = 0$ units $\Rightarrow C(0) = 7000 + 600(0) =$ $7,000.

- $q = 10$ units $\Rightarrow C(10) = 7000 + 600(10) = 7000 + 6000 = \$13,000$.

- $q = 50$ units $\Rightarrow C(50) = 7000 + 600(50) = 7000 + 30,000 = \$37,000$.

(c) Find the equation of the straight line that has slope $m = 4$ and passes through the point $(–1, 6)$.

Solution (step-by-step):

Use point-slope form: $y – y1 = m(x – x1)$.

Here $m = 4$, $(x1, y1) = (–1, 6)$.

$y – 6 = 4(x – (–1)) = 4(x + 1)$.

$y – 6 = 4x + 4$.

$y = 4x + 4 + 6 = 4x + 10$.

Answer: $y = 4x + 10$.

-- End of Assignment --